AN EXTENSION OF THE BEURLING-NEVANLINNA PROJECTION THEOREM

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ABSTRACT. We prove a polarization and a projection result for harmonic measure on the unit disk. The projection result is a generalization of the Beurling-Nevanlinna projection theorem.

1. INTRODUCTION

Let D be a domain in the complex plane $\mathbb{C}$ and let E be a Borel set on the boundary ∂D of D . The harmonic measure of E at $z \in D$ relative to D is the Perron solution $u(z)$ of the Dirichlet problem in D with boundary values 1 on E and 0 on $\partial D \setminus E$. More precisely, let $\chi_E = 1$ on E and $\chi_E = 0$ on $\partial D \setminus E$. Then

$$u(z) = \sup\{v(z) : v \text{ subharmonic in } D \text{ and } \limsup_{w \rightarrow \zeta} v(w) \leq \chi_E(\zeta) \text{ for } \zeta \in \partial D\}.$$

We will use the notation $u(z) = \omega(z, E, D)$. Some more notation: For $A \subset \mathbb{C}$, $\bar{A} = \{\bar{z} : z \in A\}$. If F is a closed set in $\mathbb{C}$, then $F_+ = F \cap \{z : \Im z \geq 0\}$, $F_- = F \cap \{z : \Im z \leq 0\}$, $F_l = F \cap \{z : \Re z \leq 0\}$ and $F_r = F \cap \{z : \Re z \geq 0\}$. If O is an open set in $\mathbb{C}$, then $O_+ = O \cap \{z : \Im z > 0\}$, $O_- = O \cap \{z : \Im z < 0\}$, $O_l = O \cap \{z : \Re z < 0\}$ and $O_r = \{z : \Re z > 0\}$.

In 1933 Beurling[2] and independently R.Nevanlinna[3] proved a projection theorem which is now a classical result for harmonic measure.

Theorem 1 (The Beurling-Nevanlinna projection theorem). *Let K be a closed set in the unit disk $\mathbb{D}$. Let $K^* = \{-|z| : z \in K\}$. Then*

$$(1.1) \quad \omega(z, \partial\mathbb{D}, \mathbb{D} \setminus K) \leq \omega(|z|, \partial\mathbb{D}, \mathbb{D} \setminus K^*), \quad \forall z \in \mathbb{D} \setminus K.$$

We will prove the following generalization of theorem 1.

Theorem 2. *Let $y \in (0, 1)$ and let K, K^* be as in theorem 1. Then*

$$(1.2) \quad \omega(iy, \partial\mathbb{D}, \mathbb{D} \setminus K) + \omega(-iy, \partial\mathbb{D}, \mathbb{D} \setminus K) \leq \omega(iy, \partial\mathbb{D}, \mathbb{D} \setminus K^*) + \omega(-iy, \partial\mathbb{D}, \mathbb{D} \setminus K^*).$$

The proof of theorem 2 is based on *polarization*.

Definition 1. Let A be a closed or open set. The polarization $\mathbb{P}A$ of A (with respect to $\mathbb{R}$) is $\mathbb{P}A = (A \cup \bar{A})_+ \cup (A \cap \bar{A})_-$.

We define also the polarization $\mathbb{P}_l A$ of A with respect to any oriented line l .

Definition 2. Let l be an oriented line and let $T_l : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ be the Möbius transformation that maps ∞ to ∞ and l to $\mathbb{R}$ preserving orientation. The polarization $\mathbb{P}_l A$ of A with respect to l is $\mathbb{P}_l A = T_l^{-1} \mathbb{P} T_l A$.

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The behaviour of harmonic measure under polarization has been studied by Solynin[5] and Betsakos[1]. One of their results is the following:

Theorem 3 (Solynin, Betsakos). *Let K be a compact set in $\mathbb{D}$, and let $z \in \mathbb{D} \setminus K$. Then*

$$(1.3) \quad \omega(z, \partial\mathbb{D}, \mathbb{D} \setminus K) + \omega(\bar{z}, \partial\mathbb{D}, \mathbb{D} \setminus K) \leq \omega(z, \partial\mathbb{D}, \mathbb{D} \setminus \mathbb{P}K) + \omega(\bar{z}, \partial\mathbb{D}, \mathbb{D} \setminus \mathbb{P}K).$$

We will prove a partial extension of theorem 3.

Theorem 4. *Let $z = re^{i\theta}$, $z' = re^{i(\theta-\pi)}$, $r \in (0, 1)$, $\theta \in [\pi/2, \pi]$. Let K be a compact set in $\mathbb{D}$ such that $K = K_1 \cup K_2 \cup K_3 \cup \overline{K_3}$, where $K_1 \subset \mathbb{D}_+$, $K_3 \subset \mathbb{D}_+$, $K_2 \subset \mathbb{D}_-$ and $\overline{K_1} \cap K_2 = \emptyset$. Assume $K_1 \subset \mathbb{D}_l$ and $K_3 \subset \mathbb{D}_r$. Then*

$$(1.4) \quad \omega(z, \partial\mathbb{D}, \mathbb{D} \setminus K) + \omega(z', \partial\mathbb{D}, \mathbb{D} \setminus K) \leq \omega(z, \partial\mathbb{D}, \mathbb{D} \setminus \mathbb{P}K) + \omega(z', \partial\mathbb{D}, \mathbb{D} \setminus \mathbb{P}K).$$

In section 2 we prove theorem 4 and see how it implies theorem 2.

2. PROOF OF THE THEOREMS

Proof of theorem 4: Let $D = \mathbb{D} \setminus K$ and $\widehat{D} = \mathbb{P}D = \mathbb{D} \setminus \overline{\mathbb{P}K}$. By the strong Markov property of harmonic measure (see [1]) we have

$$(2.1) \quad \omega(z, \partial\mathbb{D}, D) = \omega(z, (\partial\mathbb{D})_+, D_+) + \int_{-1}^1 \omega(z, dx, D_+) \omega(x, \partial\mathbb{D}, D).$$

$$(2.2) \quad \omega(z', \partial\mathbb{D}, D) = \omega(z', (\partial\mathbb{D})_-, D_-) + \int_{-1}^1 \omega(z', dx, D_-) \omega(x, \partial\mathbb{D}, D).$$

$$(2.3) \quad \omega(z, \partial\mathbb{D}, \widehat{D}) = \omega(z, (\partial\mathbb{D})_+, \widehat{D}_+) + \int_{-1}^1 \omega(z, dx, \widehat{D}_+) \omega(x, \partial\mathbb{D}, \widehat{D}).$$

$$(2.4) \quad \omega(z', \partial\mathbb{D}, \widehat{D}) = \omega(z', (\partial\mathbb{D})_-, \widehat{D}_-) + \int_{-1}^1 \omega(z', dx, \widehat{D}_-) \omega(x, \partial\mathbb{D}, \widehat{D}).$$

So, because of theorem 3, to prove (1.4) it suffices to prove

$$(2.5) \quad \omega(z, (\partial\mathbb{D})_+, D_+) + \omega(z', (\partial\mathbb{D})_-, D_-) \leq \omega(z, (\partial\mathbb{D})_+, \widehat{D}_+) + \omega(z', (\partial\mathbb{D})_-, \widehat{D}_-).$$

and

$$(2.6) \quad \omega(z, I, D_+) + \omega(z', I, D_-) \leq \omega(z, I, \widehat{D}_+) + \omega(z', I, \widehat{D}_-).$$

for every interval I on $(-1, 1)$.

First we prove (2.5). Again by the strong Markov property:

$$(2.7) \quad \omega(z, (\partial\mathbb{D})_+, D_+) = \omega(z, (\partial\mathbb{D})_+, \widehat{D}_+) - \int_{K_1} \omega(z, dk, D_+) \omega(k, (\partial\mathbb{D})_+, \widehat{D}_+).$$

and

$$(2.8) \quad \omega(z', (\partial\mathbb{D})_-, \widehat{D}_-) = \omega(z', (\partial\mathbb{D})_-, D_-) - \int_{\overline{K_1}} \omega(z', dk, \widehat{D}_-) \omega(k, (\partial\mathbb{D})_-, D_-).$$

So, to prove (2.5) it suffices to prove

$$(2.9) \quad \int_{\overline{K_1}} \omega(z', dk, \widehat{D}_-) \omega(k, (\partial\mathbb{D})_-, D_-) \leq \int_{K_1} \omega(z, dk, D_+) \omega(k, (\partial\mathbb{D})_+, \widehat{D}_+).$$

But we have

$$(2.10) \quad \omega(z', \bar{J}, \hat{D}_-) \leq \omega(z, J, D_+)$$

for every Borel set J on the boundary of K . (2.10) holds because of (a) the domain monotonicity of harmonic measure, and (b) the polarization theorem for harmonic measure (theorem 2.2 in [1]). We also have

$$(2.11) \quad \omega(k, (\partial\mathbb{D})_-, D_-) \leq \omega(k, (\partial\mathbb{D})_+, \hat{D}_+).$$

by domain monotonicity. (2.10) and (2.11) show that (2.9) holds. Hence (2.5) also holds. The proof of (2.6) is very similar to the proof of (2.5) and we omit it. As we noted above (2.5) and (2.6) imply (1.4) and theorem 4 is proved.

Proof of theorem 2: To prove theorem 2 we use a technique due to Øksendal. First we note that theorem 4 and the domain monotonicity of harmonic measure imply:

$$(2.12) \quad \omega(z, \partial\mathbb{D}, D) + \omega(z', \partial\mathbb{D}, D) \leq \omega(z, \partial\mathbb{D}, \mathbb{D} \setminus \overline{\mathbb{P}K}_-) + \omega(z', \partial\mathbb{D}, \mathbb{D} \setminus \overline{\mathbb{P}K}_-).$$

We consider successive polarizations of D with respect to the line $\{re^{i\theta} : r \in \mathbb{R}, \theta = -\pi/2^n\}$ (with the orientation from $e^{-i\theta}$ to $e^{i(\pi-\theta)}$) and then with respect to the line $\{re^{i\theta} : r \in \mathbb{R}, \theta = \pi/2^n\}$ (with the orientation from $e^{i\theta}$ to $e^{i(\pi+\theta)}$), $n = 0, 1, 2, \dots$. In each step we apply (2.12). In the limit we obtain theorem 2.

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